CCSF PHYC 4D Lecture Notes

Karl M. Westerberg

Chapter 2
The Special Theory of Relativity

History (outline)

Students interested in some additional details might consider looking up "History of special relativity" in wikipedia.

The term "relativity" simply refers to the idea that all inertial reference frames are equivalent, and that there is no physical experiment that can select between the different reference frames. This idea has been around at least since Galileo (if not earlier). The Galilean transformation refers to the change of coordinates that results when the same (space-time) event(s) is/are being measured by two different observers moving relative to each other. If we suppose that the origins and spatial axes of S and S' coincide, and that the velocity of S' relative to S (denoted $\vec{u}$) point in the x direction, then

$$x' = x - ut \qquad y' = y \qquad z' = z \qquad t' = t$$

or more succinctly,

$$\vec{r}' = \vec{r} - \vec{u}t \qquad t' = t$$

The fact that $t' = t$ was taken for granted. Nobody even considered the possibility that the two observers would have a different notion of the passage of time.

Newtonian mechanics takes the Galilean transformation for granted. One of the consequences is the usual velocity addition formula: $\vec{v}' = \vec{v} - \vec{u}$. If we define A = particle, $B = S'$, and $C = S$, the velocity addition formula can be written in a more familiar form: $\vec{v}_{AB} = \vec{v}_{AC} - \vec{v}_{BC}$, or equivalently, $\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$.

First clue of the failure of Galilean relativity: Biot-Savart Law.

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{s} \times \hat{r}}{r^2}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2}$$

Consider two positive charges $+q$, distance r apart, and both moving with velocity $\vec{v}$ perpendicular to separation of charges. Net force on top charge (including electrical) is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}(1 - \epsilon_0\mu_0 v^2)\hat{r}$$

An observer also moving with velocity $\vec{v}$ will not observe the magnetic force ($v = 0$), and will record a different force, and (more importantly), will observe different motion.

Second clue: Maxwell's equations. [This would be a good opportunity to state Maxwell's equations in differential form and derive the wave solution.]

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla} \times \vec{E} = -\partial\vec{B}/\partial t$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} - \epsilon_0\mu_0\partial\vec{E}/\partial t = \mu_0\vec{J}$$

Ansatz:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \sin(\vec{k} \cdot \vec{r} - \omega t) \qquad \vec{B}(\vec{r}, t) = \vec{B}_0 \sin(\vec{k} \cdot \vec{r} - \omega t)$$

Then Maxwell's equations (with $\rho = 0$ and $\vec{J} = 0$) gives

$$\vec{k} \cdot \vec{E}_0 = 0$$

$$\vec{k} \times \vec{E}_0 = +\omega\vec{B}_0$$

$$\vec{k} \cdot \vec{B}_0 = 0$$

$$\vec{k} \times \vec{B}_0 + \epsilon_0\mu_0\omega\vec{E}_0 = 0$$

Directionally, this implies that $\vec{k}$ (which points in the direction that the wave travels), $\vec{E}_0$, and $\vec{B}_0$ are mutually perpendicular, with $\vec{k} \to \vec{E}_0 \to \vec{B}_0$ right-handed (e.g., $\vec{k}$ in z direction, $\vec{E}_0$ in x direction, $\vec{B}_0$ in y direction). In terms of magnitudes: $E_0 = \omega/kB_0 = cB_0$, and $B_0 = \epsilon_0\mu_0(\omega/k)E_0 = \epsilon_0\mu_0 cE_0$. This implies that $\epsilon_0\mu_0c^2 = 1$, or that $c = 1/\sqrt{\epsilon_0\mu_0}$.

As just derived, one of the consequences of Maxwell's equations is that an electromagnetic wave can travel with speed $c = 1/\sqrt{\epsilon_0\mu_0}$. The assertion that this result be true in all inertial reference frames is inconsistent with the Galilean transformation [rocket example].

Maxwell, perhaps recognizing this, but also unwilling to accept that a wave can travel through free space without a medium (consider sound waves, and waves on a string — a medium is always involved), designated that these

waves travel through an invisible *ether*. In fact, one is forced to conclude that Maxwell's equations themselves can only be valid in a reference frame in which the ether is at rest.

Physicists were trying to measure various properties of this ether, which mysteriously seemed to avoid detection. In 1887, Michaelson and Morley conducted an experiment designed to measure the speed of the earth relative to the ether by comparing the speed of light in different directions (more later). (Michaelson had done an earlier version of this experiment but overestimated the sensitivity of the experiment — no conclusion could be drawn.) They concluded that the earth travelled less than 5 km/s (Krane, p. 30[25]), which is much less than the 30 km/s that the earth travels relative to the sun. Repeating the experiment 6 months later should give different results. It did not. One is led to conclude that either there is no ether (and Galilean relativity must be replaced by something else), or else the ether is dragged along with the earth somehow, so we are always at rest with respect to the ether.

Meanwhile, in 1887, Heinrich Hertz demonstrated the existance of electromagnetic waves. People pretty much accept Maxwell's equations at this point.

Lorentz developed the Lorentz transformation in an attempt to reconcile the negative result of the Michaelson-Morley experiment with other experimental evidence.

$$x' = \gamma(x-ut) \qquad y' = y \qquad z' = z \qquad t' = \gamma(t-(u/c^2)x) \qquad \gamma = 1/\sqrt{1-u^2/c^2}$$

$$\vec{r}'_{\parallel} = \gamma(\vec{r}_{\parallel} - \vec{u}t) \qquad \vec{r}'_{\perp} = \vec{r}_{\perp} \qquad t' = \gamma(t - \vec{u}\cdot\vec{r}/c^2)$$

(As it turns out, Maxwell's equations are invariant under the Lorentz transformation, although it apparently took him quite some time to prove this.) He denoted the new time coordinate as local time. Poincare took Lorentz's work and extended it, noting that "local time" can be measured by clocks that can be synchronized by light pulses. A lot of the arguments that Einstein later used were developed earlier by other physicists, including Lorentz and Poincare. It seemed that the physicists of the day were beginning to realize the consequences of relativity: that lengths contract, time is dialated, synchronization is relative, and masses increase with velocity (in a manner

we discuss later). As advanced as Poincare's ideas became, he and others still held fast to the notion of an ether, upon which reality is based (they would often make statements like "it is possible that an observer might observe events A and B to be simultaneous, even though, in reality, they are not."). It was Einstein who combined all of these results, and abolished the notion of ether once and for all.

Michaelson-Morley experiment

The Galilean transformation implies that Maxwell's equations can only be valid in one reference frame, the reference frame where the ether is at rest. If the ground (earth) moves relative to this ether ($\vec{u} = \vec{v}_{GE}$), then the speed of light relative to the ground ($\vec{v} = \vec{v}_{LG}$) will not be constant; it will depend on the angle between between $\vec{v}_{LG}$ and $\vec{v}_{GE}$. [Ch2;p1]

Indeed, v can be computed in terms of u, c, and the angle ϕ between $\vec{v}$ and $\vec{u}$.

$$(v + u\cos\phi)^2 + u^2\sin^2\phi = c^2$$
$$v + u\cos\phi = \sqrt{c^2 - u^2\sin^2\phi}$$
$$v = \sqrt{c^2 - u^2\sin^2\phi} - u\cos\phi$$

Three special cases:

$$\phi = 0^\circ \rightarrow v = c - u$$
$$\phi = 180^\circ \rightarrow v = c + u$$
$$\phi = 90^\circ \rightarrow v = \sqrt{c^2 - u^2}$$

[Ch2;p1: Picture of interferometer: describe it]

Let N_1 and N_2 be the number of wavelengths of light travelling along paths 1 and 2, respectively. When the two waves meet at the screen, they interfere according to the fractional part of the difference $\Delta N = N_2 - N_1$. If the difference is a whole integer, the interference is constructive and a bright spot appears on the screen. If ΔN is a whole integer + 1/2, then the interference is destructive and a dark spot appears.

What really happens is that multiple light rays pass through the apparatus with slightly different path lengths (e.g., suppose one or both mirrors are tilted slightly), and the result is a series of bright and dark fringes on the

screen (interference is constructive or destructive depending on where the light strikes the screen). You will be using an interferometer in PHYC 4DL.

Suppose $\vec{v}_{GE}$ lines up with path 1. Then

$$N_1 = \frac{L_1}{(c-u)/\nu} + \frac{L_1}{(c+u)/\nu} = \frac{2L_1\nu}{c}\gamma^2 = \frac{2L_1}{\lambda_0}\gamma^2$$

$$N_2 = \frac{L_2}{\sqrt{c^2-u^2}/\nu} \times 2 = 2\frac{L_2\nu}{c}\gamma = \frac{2L_2}{\lambda_0}\gamma$$

where $\gamma = 1/\sqrt{1-u^2/c^2}$ and $\lambda_0 = c/\nu$. Notice the subtle change in power of γ between the two cases. This is related to the fact that the speed of light depends on direction. In this case,

$$\Delta N = N_2 - N_1 = \frac{2\gamma(L_2 - \gamma L_1)}{\lambda_0}$$

Okay, so we have a value of ΔN which results in a series of bright and dark fringes. So what?

Well, what happens if ΔN is changed in some way. In PHYC 4DL, you will modify ΔN in two different ways: (1) you can change L_1 and/or L_2 by moving the mirror(s) back and forth, and (2) you can change the wavelength of light along one of the paths by changing the index of refraction through part of that path (you will be measuring the index of refraction of air this way). As ΔN changes, the pattern of bright and dark fringes *shifts*. Each time the fringes shift over by 1, that represents a change in ΔN by 1. This is a very sensitive apparatus, in that shifting ΔN by 1 does not take very much since the wavelength of visible light is really short (e.g., changing L_1 by half a wavelength results in ΔN changing by 1).

In the Michaelson-Morley experiment, ΔN is changed by rotating the apparatus. If you rotate the apparatus by 90°, path 2 will line up $\vec{v}_{GE}$ instead of path 1 and the powers of γ will switch. After the rotation,

$$\Delta N = N_2 - N_1 = \frac{2\gamma(\gamma L_2 - L_1)}{\lambda_0}$$

The change in ΔN (written $\Delta\Delta N$) is given by

$$\Delta\Delta N = \frac{2(L_1 + L_2)}{\lambda_0}\gamma(\gamma - 1)$$

Note that if u is much less than c, then $\gamma - 1$ is approximately $\frac{1}{2}u^2/c^2$, which is quite small, so this apparatus needs to be very sensitive. One way to increase the sensitivity of this apparatus is to use a large value of $L_1 + L_2$ (although one has to worry about vibrations when $L_1 + L_2$ is large). For a given value of u, how big does $L_1 + L_2$ need to be in order for ΔN to change by 1? Plug in the approximate values for γ and $\gamma - 1$:

$$\Delta\Delta N = \frac{2(L_1 + L_2)}{\lambda_0}(1)\frac{u^2}{2c^2}$$

Solve for $L_1 + L_2$:

$$L_1 + L_2 = (\Delta\Delta N)\lambda_0 c^2/u^2$$

For visible light $\lambda_0 = 500\,\mathrm{nm}$ and $u = 30\,\mathrm{km/s}$ (the speed of the earth relative to the sun), $L_1 + L_2 = 50\,\mathrm{m}$. The PHYC 4DL apparatus is not large enough to perform this experiment.

When Michaelson and Morley performed this experiment, they had an effective $L_1 + L_2$ of about $40\,\mathrm{m}$ (?) and were able to determine through careful measurement that $\Delta\Delta N < 0.01$. They concluded that $u < 5\,\mathrm{km/s}$.

Technical issue: how do we know $\vec{v}_{GE}$ lines up with path 1 initially? We don't. What happens is that ΔN depends on the angle between $\vec{v}_{GE}$ and path 1. The calculation for general angle is very complicated, but the result is that the two extreme values of ΔN occur when that angle is $0°$ (or $180°$) and $90°$. So the apparatus is rotated slowly, and fringe motion is observed. It is expected that the variation of ΔN over the entire range of angles will be as we had calculated above for $0°$ and $90°$.

What if $\vec{v}_{GE}$ lies outside the plane of the apparatus? Stop asking hard questions...

Einstein postulates and their consequences

Einstein's theory of special relativity is founded upon two postulates:

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light in free space (vacuum) is the same in all inertial reference frames.

In light [sic] of Maxwell's equations, the first postulate implies the second one, but we still state them separately.

[Have two meter sticks and two clocks available]

There are three major kinematic consequences of Einstein's postulates:

1. Time dilation: moving clocks run slower than stationary clocks. More precisely, an observer for whom a given clock is moving will observe that clock to run slower than another observer for whom the given clock is stationary. ALT: If you have two clocks, one moving and the other at rest, that are otherwise identical, then the one that is moving will run slower than the one at rest.

2. Length contraction: moving objects are physically shorter along the direction of motion than stationary objects are. More precisely, an observer for whom a given object is moving will measure a shorter length along the direction of motion than another observer for whom the given object is at rest. Lengths measured *perpendicular* to the relative motion of the two observers are unaffected — both observers measure the same lateral lengths.

3. This is a good place to talk about the pole vaulter paradox.

4. Relativity of simultaneity: two events that are simultaneous relative to one observer may not be simultaneous relative to another observer. (Note the subtle difference between this statement and the hypothetical statement I attributed to Poincare earlier: Einstein would not have claimed that either observer was wrong in their assessment of simultaneity; rather he called into question whether simultaneity is an absolute concept, independent of observer: he stated that it is not.)

[Note: Time dilation and length contraction can each be applied to two otherwise identical clocks/objects, one moving and one not relative to the *same* observer, or equivalently, to a single clock/object observed by two different observers making time/length measurements.]

Einstein derived these consequences with the use of *thought experiments*, which are experiments that he didn't actually carry out, but he *imagined* them to be carried out in a universe where his two postulates were true. His

derivations made extensive use of light pulses, since the constancy of c was such a central feature of his relativity theory.

Time dilation

- Moving clocks run slower.
- Lateral clock: length L. photon moves up and down in time 2L/c (one tick) [Ch2;p2].
- In frame where clock moves $+u\vec{\imath}$:

$$c^2 t_1^2 = L^2 + d^2 = L^2 + u^2 t_1^2$$

$$t_1 = \frac{L\gamma}{c} = t_2$$

$$\text{total time} = t_1 + t_2 = \frac{2L}{c}\gamma = \gamma\tau$$

- All other types of clocks also run slow by same rate: if they didn't, we could distinugush reference frames based on agreement between different types of clocks.
- Muon lifetime ($\tau = 2.2\,\mu\text{s}$, $c\tau = 660\,\text{m}$). Ex 2.4 in textbook calculates minimum u required for muon to survive 100 km in atomosphere: $\gamma >= 100\,\text{km}/660\,\text{m} = 150$.

Length contraction

- Moving objects are shorter than they are when stationary.
- Orient photon clock along direction of motion (length L). When clock at rest, photon goes back and forth in time $2L/c$ (one clock tick) [Ch2;p3].
- When clock moves $+u\vec{\imath}$: On the way out...

$$d_1 = L' + ut_1 = ct_1$$

On the way back...

$$d_2 = L' - ut_2 = ct_2$$

$$L' = (c-u)t_1 = (c+u)t_2$$

$$\text{Total time} = L'/(c-u) + L'/(c+u) = \frac{2L'}{c}\gamma^2 = \frac{2L}{c}\gamma$$

So $L' = L/\gamma$.

- Note similarity to Michaelson-Morley expiment. In fact, length contraction was considered a possible explanation of negative result.
- Reconsider muon lifetime.

Relativity of Simultaneity (qualitative)

- S at rest (in S frame) in middle of cart. Two events A and B occur simultaneously at both ends. Light from events reach S at same time [Ch2;p4].
- S' moving $u\vec{\imath}$ relative to S. S and S' coincide at moment that A and B occur (according to S).
- If A and B simultaneous in S' as well, then both photons should reach S' simultaneously too (since S' is at midpoint when both events occur).
- Not possible. S' is moving towards B (to right), so that photon reaches S first. B occurred first according to S'.
- NOTE: This depends on speed of light always being equal to c. Under the Galilean transformation, the fact that B's photon reaches S' first is explained by the fact that light must travel at different speeds relative to S'. Simultaneity is forced.
- If S' moves $-u\vec{\imath}$ relative to S, then A occurs first for S'.
- Lack of agreement about simultaneity does not violate the principle of causality. Speed of any signal must be $\leq c$, always.

Lateral length non-adjust

- Two meter sticks oriented along y-axis. Stick B moves $+u\vec{\imath}$ relative to stick A.

- Thumb tacks on ends: shorter stick scratches longer stick.
- Stick A moves $-u\vec{\imath}$ relative to stick B. After 180° rotation, velocity becomes $+u\vec{\imath}$, and the roles played by A and B reverse.
- It follows that if stick B sees stick A as longer, then stick A must see stick B as longer (and vice versa).
- Impossible because of scratches.
- This argument fails for longtitudinal orientation, because both meter sticks would be scratched all the way across, no matter how long each is relative to the other.

Illustration (2 slides and handout)

- A reference frame is defined by a ruler with tick marks (each tick corresponds to one length unit) and a series of clocks, one per tick, that are synchronized *in that reference frame.* Each tick on the clock corresponds to one time unit.
- Two reference frames are assumed to exist, S and S'. The two frames are lined up at $(x, t) = (0, 0)$ and $(x', t') = (0, 0)$. S' moves to the right with respect to S at a speed $u = 1\,\mathrm{Lu/Tu}$. $\gamma = 1/\sqrt{1 - u^2/c^2}$ is assumed to be 2.
- One page (the "S page") shows how the rulers and clocks line up with each other with respect to the S reference frame. Note that each snapshot occurs at a constant value of t, with $x = 0$ always in the middle. Events with x and t each ranging from -2 to $+2$ (integer values) are shown.
- The other page (the "S' page") shows how the rulers and clocks line up with each other with respect to the S' reference frame. Note that each snapshot occurs at a constant value of t', with $x' = 0$ always in the middle. Events with x' and t' each ranging from -2 to $+2$ (integer values) are shown.
- The S clock at $x = 0$ and S' clock at $x' = 0$ are both shown larger than all of the other clocks on both pages.

- The following event points will be discussed below. It might be a good idea to highlight them ahead of time.

event	(x,t)		(x',t')	on S page?	on S' page
A	$(0,0)$	$\leftrightarrow$	$(0,0)$	Y	Y
B	$(+1,+1)$	$\leftrightarrow$	$(0,+\frac{1}{2})$	Y	N
C	$(+2,+2)$	$\leftrightarrow$	$(0,+1)$	Y	Y
D	$(+2,+1\frac{1}{2})$	$\leftrightarrow$	$(+1,0)$	N	Y
E	$(0,+1)$	$\leftrightarrow$	$(-2,+2)$	Y	Y
F	$(+2,+1)$	$\leftrightarrow$	$(+2,-1)$	Y	Y

- Note that, according to S, the S' clocks are *not* synchronized, and vice versa. This illustrates the relativity of simultaneity. According to S, the two events E$(0,1)$ and F$(2,1)$ are simultaneous, but according to S', they are not: E$(-2,2)$ and F$(2,-1)$. Note that S' moves to the right with respect to S, so that the rightmost event (F) occurs earlier in the S' frame.

- Now compare clock speeds. According to S, the S' clocks should run slow. They do. Compare events A, B, and C, as we follow the big $x'=0$ clock of S'. That clock ticks a half unit each time one unit of time passes in the S frame. Note that a similar thing happens with the big S clock in the S' frame (the big S clock appears to be moving at half speed according to the S' clocks).

- Note that the events A, B, and C occur in *different* places for S since the big S' clock is moving with respect to S. Thus, the S observer is using different clocks to track the progress of the big S' clock. This is not a problem for S, since he believes that his clocks are all synchronized.

- When S makes this claim to S' and uses events A, B, and C as evidence that the S' clocks are running slower, S' will complain. Consider events A and C as shown on the S' page. According to S', at the same time S reads the big S' clock at $x=t=0$ (event A), the S clock located at $x=2$ has already advanced to $t=1\frac{1}{2}$ (this is event D, which is simultaneous to A according to S'). So when the $x=2$ clock finally reaches $x'=0$ at $t'=1$ (event C), naturally that clock will have advanced to $t=2$. But, according to S', that clock only advanced a half time unit, not two time units as claimed by S.

- Compare the separation of the tick marks for the two rulers on each page. Note that each observer observes the other frame's tick marks to be closer together by a factor of two. This is an illustration of length contraction.

- The thick ruled line in each diagram represents an object that is at rest in the S frame. According to S, this object has a (proper) length of 2 units. The same object on the S' page appears to be moving to the left and has a length of 1 unit, due to length contraction.

- Looking at this object depicted on the S' page, the length comparison seems rather evident. The same object spans one tick mark on the S' ruler and two tick marks on the S ruler.

- Examining this object on the S page, however, yields an apparent conflict: the object clearly spans two tick marks on the S ruler, but appears to span *four* tick marks on the S' ruler, in apparent contradiction to its length equalling one length unit in that frame.

- One must be careful in interpreting this. Consider events E and F as shown on the S page as an example. Those two events are simultaneous in the S frame with $\Delta x = 2$, the proper length of the object. They are *not* simultaneous in the S' frame however. Examining those two events depicted on the S' page reveals the issue: yes E corresponds to the left side of the object and F corresponds to the right side of the object, and the two events are separated in space by $\Delta x' = 4$. But F occurs 3 time units earlier than E. During that time, the object will have moved 3 length units to the left, accounting for the discrepancy between $\Delta x'$ between the two *events* and the actual length of the object as measured by S'.

- By contrast, look at events A and D on the S' page. Those events are simultaneous in S' and represent an honest measurement of the length in the S' frame. Those same two events are *not* simultaneous in S, but that does not matter, since the object does not move relative to S.

- One final point: note that through all of these examples, that both observers agree as to the *identity* of the events, and the correspondance

between (x,t) and (x',t'). Events A, C, E, and F appear on both pages. In each case, the correspondance between (x,t) and (x',t') is the same.

How to synchronize clocks

- Two clocks A and B, at rest, distance L apart. Each clock emits a photon towards the other.
- Photon 1: A (t_1^A) to B (t_1^B). Photon 2: B (t_2^B) to A (t_2^A).
- If clocks are already synchronized, then

$$t_1^B - t_1^A = L/c = t_2^A - t_2^B$$

$$t_1^B + t_2^B = t_1^A + t_2^A$$

- If clocks are not synchronized, then we need to adjust clock A by Δt, so that

$$t_1^B + t_2^B = (t_1^A + \Delta t) + (t_2^A + \Delta t) = (t_1^A + t_2^A) + 2\Delta t$$

  Solve for Δt:

$$\Delta t = \frac{1}{2}[(t_1^B + t_2^B) - (t_1^A + t_2^A)]$$

Lorentz Transformation

- Spacetime events are described by (t,x,y,z) coordinates by each observer. Coordinates for the same spacetime event as viewed by two different observers (S' moving relative to S with velocity $+u\vec{\imath}$) are related by the Lorentz transformation:

$$x' = \gamma(x - ut) \qquad y' = y \qquad z' = z \qquad t' = \gamma(t - (u/c^2)x) \qquad \gamma = 1/\sqrt{1 - u^2/c^2}$$

- Spatial and temporal separation between two events, as observed by different observers, are also related by the Lorentz transformation:

$$\Delta x' = \gamma(\Delta x - u\Delta t) \qquad \Delta y' = \Delta y \qquad \Delta z' = \Delta z \qquad \Delta t' = \gamma(\Delta t - (u/c^2)\Delta x)$$

- Time dilation: if S observes two events to occur at the same place ($\Delta x = 0$), then
$$\Delta t' = \gamma(\Delta t - (u/c^2)\Delta x) = \gamma \Delta t$$
Successive ticks of a stationary clock occur at the same place, so this is time dilation.

- If an object oriented along x is at rest in S', then its length can be determined by two events, not simultaneous in S', but in S where the object is moving (so $\Delta t = 0$):
$$L_0 = \Delta x' = \gamma(\Delta x - u\Delta t) = \gamma \Delta x = \gamma L$$
so $L = L_0/\gamma$. Note: $\Delta t' \neq 0$, but who cares?

- Two events simultaneous in S, but spatially separated (in x direction) are not simultaneous in S':
$$\Delta t' = \gamma(\Delta t - (u/c^2)\Delta x) = -\gamma(u/c^2)\Delta x$$
Note minus sign: higher x implies lower t' (assuming $u > 0$). Event that S' is moving towards occurs earlier to S'.

Velocity addition

- If S and S' observe a moving object, then
$$v'_x = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - u\Delta t)}{\gamma(\Delta t - (u/c^2)\Delta x)} = \frac{v_x - u}{1 - uv_x/c^2}$$
Note if u and/or v is significantly less than c, this reduces to $v'_x = v_x - u$ (GT).

- Calculations:
$$\begin{array}{ccc} u = -0.1c & v_x = 0.1c & v'_x = 0.2c/1.01 = 0.198c \\ u = -0.6c & v_x = 0.6c & v'_x = 1.2c/1.36 = 0.882c \\ u = -0.9c & v_x = 0.9c & v'_x = 1.8c/1.81 = 0.994c \\ u = \text{anything} & v_x = c & v'_x = (c-u)/(1-u/c) = c \end{array}$$
No matter what, $v'_x \leq c$.

- v_y and v_z likewise transform: not identity because $\Delta t'$ is different from Δt
$$v'_y = \frac{\Delta y'}{\Delta t'} = \frac{v_y}{\gamma(1 - uv_x/c^2)}$$
- Calculation involving two dimensions — read the book.

Invariant interval

- Invariant interval ($(\Delta s)^2$) between two events:

  $$c^2(\Delta t')^2 - (\Delta x')^2 = c^2\gamma^2(\Delta t - (u/c^2)\Delta x)^2 - \gamma^2(\Delta x - u\Delta t)^2 = c^2(\Delta t)^2 - (\Delta x)^2$$

  Value is independent of observer. Note: add $(\Delta y)^2$ and $(\Delta z)^2$ at will.
- $(\Delta s)^2 = (c\tau)^2 > 0$: time-like separation. Cannot be simultaneous in any frame. If $\Delta t > 0$ in one frame, then $\Delta t > 0$ in all frames and event 1 occurs before event 2, and can cause event 2. τ is the time measured by an observer who sees the two events happening at the same place (proper time). All other observers observe longer time intervals than τ.
- $(\Delta s)^2 = -d^2 < 0$: space-like separation. Cannot occur at the same point in space. d is the distance between the two events as measured by an observer who sees the events as simultaneous. All other observers observe $|\Delta\vec{r}|$ to be larger than d, and can observe $\Delta t < 0$ or $\Delta t > 0$ (either event can occur first), but no signal can travel between the two events (neither event can cause the other), since all frames observe $|\Delta\vec{r}|/\Delta t > c$.
- Note: d (above) should not necessarily be associated with the *proper length* of an object. You would not use non-simultaneous events to measure the object's length in a different frame where the object is moving.
- $(\Delta s)^2 = 0$: null separation or light-like separation. Null separated events can occur along the space-time path of light. They cannot be brought into coincidence, but $|\Delta\vec{r}|$ and Δt can be made arbitrarily small by appropriate choice of frame. Note that $|\Delta\vec{r}|/\Delta t = c$ in all frames.

Doppler effect — Classical

- Wave emitted by S (source), received by T (target). All quantities below are relative to the medium.

$$\begin{aligned}
v_W &= \text{Velocity of wave (+x direction)} \\
v_S &= \text{Velocity of source} \\
v_T &= \text{Velocity of target} \\
\lambda &= \text{Wavelength (distance between wavefronts)} \\
\Delta t_S &= \text{Time between wavefronts emitted by source} \\
\Delta t_T &= \text{Time between wavefronts received by target}
\end{aligned}$$

- If S and T at rest, then $\lambda = v_W \Delta t_S = v_W \Delta t_T$, and so

$$\nu_T/\nu_S = \Delta t_S/\Delta t_T = 1$$

- If S is moving (1-D case only; see [Ch2:p5]):

$$v_W \Delta t_S = v_S \Delta t_S + \lambda \qquad \lambda = (v_W - v_S)\Delta t_S$$

- If T is moving:

$$\lambda = (v_W - v_T)\Delta t_T$$

- Put it together:

$$\nu_T/\nu_S = (v_W - v_T)/(v_W - v_S) = (1 - v_T/v_W)/(1 - v_S/v_W)$$

- If S moves towards T ($v_S > 0$), then wavefronts bunch together, λ decreases and ν_T increases. If T moves towards S ($v_T > 0$), T encounters wavefronts more quickly, and ν_T increases.

- Alternative derivation:

$$v_{WS} = \lambda \nu_S \qquad v_{WT} = \lambda \nu_T \qquad \nu_T/\nu_S = v_{WT}/v_{WS}$$

- More than one dimension:

$$\nu_T/\nu_S = (v_W - \hat{n} \cdot \vec{v}_T)/(v_W - \hat{n} \cdot \vec{v}_S)$$

  where $\hat{n}$ is a unit vector pointing from where S is when it emits a wavefront to where T *will be* when it receives that same wavefront.

- Examples: speed of sound in air $v_W = 343\,\mathrm{m/s}$.

$$v_S = 0 \quad v_T = -24.5\,\mathrm{m/s} = -(1/14)v_W$$

$$\nu_T/\nu_S = (1 + 1/14)/(1) = 15/14 = 1.0714$$

$$v_S = +24.5\,\mathrm{m/s} = +(1/14)v_W \quad v_T = 0$$

$$\nu_T/\nu_S = 1/(1 - 1/14) = 14/13 = 1.0769$$

  Difference is attributed to the medium.

Doppler effect — Relativistic

- All quantities in classical derivation are relative to one reference frame (the medium), so no change up to this point:

$$\lambda = (v_W - v_S)\Delta t_S = (v_W - v_T)\Delta t_T$$

- Note that $\Delta t_{S,T}$ are measured by clocks at rest in the *medium*, whereas $\nu_{S,T}$ are defined by clocks which are at rest relative to S and T, respectively. The latter times ($\Delta t_{S0,T0}$) are *proper times*.

- Put it together:

$$\nu_T/\nu_S = \Delta t_{S0}/\Delta t_{T0} = (\gamma_T/\gamma_S)\Delta t_S/\Delta t_T = (\gamma_T/\gamma_S)(1-v_T/v_W)/(1-v_S/v_W)$$

  Note: alternative classical derivation doesn't work so well.

- Examples: speed of light in a medium $v_W = c/n = \frac{4}{5}c$ (if $n = \frac{5}{4}$).

$$v_S = 0 \quad v_T = -\frac{3}{5}c = -\frac{3}{4}v_W \quad (\gamma_T = \frac{5}{4})$$

$$\nu_T/\nu_S = [(5/4)/1][(1 + \frac{3}{4})/1] = 35/16 = 2.1875$$

$$v_S = +\frac{3}{5}c = +\frac{3}{4}v_W \quad (\gamma_S = \frac{5}{4}) \quad v_T = 0$$

$$\nu_T/\nu_S = [1/(5/4)][1/(1 - \frac{3}{4})] = 16/5 = 3.2000$$

  Again, difference is attributed to medium.

- Now suppose $v_W = c$. Then

$$\nu_T/\nu_S = \sqrt{\frac{1-v_T/c}{1+v_T/c}}\sqrt{\frac{1+v_S/c}{1-v_S/c}}$$

For either $(v_S, v_T) = (0, -u)$ or $(+u, 0)$

$$\nu_T/\nu_S = \sqrt{(1+u/c)/(1-u/c)}$$

Medium does not matter. General case (which involves velocity addition formula) is left as an exercise.

Relativistic Dynamics

- Example: Infinite plane $\sigma = -1.00\,\mu\mathrm{C/m}^2$, electron located on right side and accelerates from rest in $+x$ direction. After $10^{-7}\,\mathrm{s}$, how fast is the electron moving?

$$\epsilon_0 = 8.85\,\mathrm{pF/m} \qquad e = 1.60 \times 10^{-19}\,\mathrm{C} \qquad m = 9.11 \times 10^{-31}\,\mathrm{kg}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\,\vec{\imath} = -56497\,\mathrm{V/m}\,\vec{\imath}$$

$$\vec{F} = (-e)\vec{E} = +9.04 \times 10^{-15}\,\mathrm{N}\,\vec{\imath}$$

$$\vec{a} = \vec{F}/m = +9.92 \times 10^{15}\,\mathrm{m/s}^2\,\vec{\imath}$$

$$\vec{v} = \vec{v}_0 + \vec{a}t = 9.92 \times 10^{8}\,\mathrm{m/s}\,\vec{\imath} = (3.3c)\,\vec{\imath}$$

- Second example: Proton starts from rest, undergoes potential difference $\Delta V = -1.00\,\mathrm{MV}$. How fast is proton moving? Answer the same question with an electron and $\Delta V = +1.00\,\mathrm{MV}$ (use the same physical setup to create the field, and allow the electron to move in the opposite direction).

$$m_p = 938\,\mathrm{MeV}/c^2 \qquad m_e = 0.511\,\mathrm{MeV}/c^2$$

$$K_f = 1.00\,\mathrm{MeV} \qquad \text{(either case)}$$

$$v_f = \sqrt{2K_f/m} = \text{(proton)}0.046c = \text{(electron)}1.98c$$

- Book shows examples of how analyzing collisions with non-relativistic dynamics leads to conservation of momentum/energy in one reference frame, and failure of conservation of momentum/energy in another reference frame.

- Failure of these examples can be fixed by changing the definition of momentum and energy:

$$\vec{F}_{\text{net}} = d\vec{p}/dt \qquad \vec{p} = \gamma m \vec{v} \qquad E = \gamma mc^2 \qquad \gamma = 1/\sqrt{1 - v^2/c^2}$$

- If $v \ll c$, then (check)

$$\vec{p} = \gamma m\vec{v} \approx m\vec{v}$$

$$E = \gamma mc^2 \approx (1 + v^2/(2c^2))mc^2 = mc^2 + \frac{1}{2}mv^2$$

  Note the addition of a new term for *rest* energy, $E_0 = mc^2$. Einstein postulated this form of energy, which plays a critical role in nuclear decays where the total mass of the system changes. It has since been verified that energy is released when mass decreases, in accordinance with this result.

- Note how definition of $\vec{p}$ and E protect speed of light limit. As $v \to c$, p and E both increase without bound. Force in example 1 can apply forever without allowing $v > c$. Likewise in example 2, energy can be supplied forever by an ever-increasing potential difference without allowing $v > c$.

- Example 1 relativistically:

$$\vec{F} = +9.04 \times 10^{-15}\,\mathrm{N}\,\vec{\imath} = d\vec{p}/dt$$

  Since $\vec{F}$ is constant

$$\Delta\vec{p} = \vec{F}\Delta t = +9.04 \times 10^{-22}\,\mathrm{N\,s}\,\vec{\imath}$$

$$\vec{p}_f = \gamma m\vec{v}_f = +9.04 \times 10^{-22}\,\mathrm{kg\,m/s}\,\vec{\imath}$$

$$\gamma v = 9.92 \times 10^{8}\,\mathrm{m/s} = 3.31c = \alpha c$$

$$v = \sqrt{\frac{\alpha^2}{1+\alpha^2}}c = 0.957c$$

- Example 2 relativistically. For proton:

$$E = \gamma mc^2 = 938\,\mathrm{MeV} + 1.00\,\mathrm{MeV} = 939\,\mathrm{MeV} = \gamma(938\,\mathrm{MeV})$$

$$\gamma = 939/938 = 1.00107$$

$$v = \sqrt{1 - 1/\gamma^2} = 0.046c \qquad \text{(same as before)}$$

For electron:

$$E = 0.511\,\text{MeV} + 1.00\,\text{MeV} = 1.511\,\text{MeV} = \gamma(0.511\,\text{MeV})$$

$$\gamma = 1.511/0.511 = 2.957 \qquad v = 0.941c$$

- An example of analyzing a collision relativistically will be given when we discuss the Comptom effect (Chapter 3).

- $$E^2 - (pc)^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2 = \gamma^2 m^2 c^4 (1 - v^2/c^2) = m^2 c^4$$
$$E^2 = (pc)^2 + (mc^2)^2$$

  In the extreme relativisitic limit, $\gamma \to \infty$, so $pc \gg mc^2$, in which case $E = pc$. This relationship holds exactly for photons (light).

- The meaning of $\vec{F}_{\text{net}} = m\vec{a}$. Suggested reading:

  Feynmann Lectures on Physics, I-12.1 (what is a force?)

  - $\vec{F}_{\text{net}} = m\vec{a}$ (N2) is not a definition — it has physical content, and can meaningfully fail.
  - N2 is not a *complete* theory by itself. It becomes a complete theory when combined with *force laws*.
  - True physical content of N2 is that these force laws will be *simple*. Using N2 as a program for studying nature leads to *simplicity*.
  - This simplicity fails when we consider relativistic speeds. The failure of the examples 1 and 2 to be consistent with nature illustrates this.
  - Einstein chose to maintain the simplicity of the force laws (preserve Coulomb's Law, and various conservation principles), and changed N2 itself by redefining $\vec{p}$ and E as given above.

Four-vectors

- Set $c = 1$. This implies that 1 s is physically equivalent to 3.00×10^8 m, and establishes a relationship between length and time measurements.

- Often quantities are measured with different units because they were historically regarded as different quantities. The connection between them is discovered later. One example of this is the unit for heat transfer (calorie). This was originally defined in terms of the specific heat of water. Its relationship to the Joule (a unit of energy) was only discovered after Joule himself discovered that heat transfer was actually an *energy* transfer and that heat and energy were the same type of quantity.

- A four-vector is a four-dimensional vector, which has a time component and spatial components:

$$\mathbf{a} = (a_t, a_x, a_y, a_z) = (a_t, \vec{a}) = a_t\hat{t} + a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$$

- Examples: position and displacement four-vectors

$$\mathbf{r} = (t, x, y, z) = (t, \vec{r})$$

$$\Delta\mathbf{r} = (\Delta t, \Delta\vec{r})$$

- If S' moves $+u\,\vec{\imath}$ relative to S, then S' will observe different components of the same four-vector (different unit vectors $\hat{x}'$ and $\hat{t}'$ are used). The components of a general four-vector transform according to the Lorentz transformation:

$$a'_x = \gamma(a_x - ua_t) \qquad a'_y = a_y \qquad a'_z = a_z \qquad a'_t = \gamma(a_t - ua_x) \qquad \gamma = 1/\sqrt{1-u^2}$$

- Not all four-component vectors are legitimate four-vectors: if the components do not transform under the Lorentz transformation, then they do not define a (unique) four-vector. Example:

$$\mathbf{v} \stackrel{?}{=} d\mathbf{r}/dt = (1, \vec{v})$$

  This vector transforms under the velocity transformation rules, which are different from the Lorentz transformation, since t depends on reference frame.

- To make velocity into a four-vector, take derivative with respect to proper time (as measured by the particle itself):

$$\mathbf{v} = d\mathbf{r}/d\tau = \gamma d\mathbf{r}/dt = (\gamma, \gamma\vec{v})$$

  This transforms according to the Lorentz transformation.

- Define the inner product (dot product) between two four vectors:

$$\mathbf{a} \cdot \mathbf{b} = a_t b_t - a_x b_x - a_y b_y - a_z b_z = a_t b_t - \vec{a} \cdot \vec{b}$$

This scalar quantity is invariant (does not depend on reference frame).

$$(\Delta \mathbf{r})^2 = \Delta \mathbf{r} \cdot \Delta \mathbf{r} = (\Delta t)^2 - \Delta \vec{r} \cdot \Delta \vec{r}$$

is the invariant interval Δs^2. Also,

$$(\mathbf{v})^2 = \gamma^2 (1 - v^2) = 1$$

- Energy-momentum four-vector:

$$\mathbf{p} = m\mathbf{v} = (\gamma m, \gamma m \vec{v}) = (E, \vec{p})$$

This four-vector transforms under the Lorentz transformation:

$$p'_x = \gamma(p_x - uE) \qquad E' = \gamma(E - up_x)$$

Note also that

$$(\mathbf{p})^2 = E^2 - p^2 = (m\mathbf{v})^2 = m^2$$

so that $E^2 = p^2 + m^2$.

- Examples from electromagnetism: $\mathbf{J} = (\rho, \vec{J})$ is the four-current. $\mathbf{A} = (\phi, \vec{A})$ is the four-potential (ϕ is the electric potential that you are familiar with).

- Example: consider a rod of uniform charge density ρ at rest (so $\vec{J} = 0$). Now transform to S' frame:

$$\rho' = \gamma \rho \qquad \vec{J}' = -\gamma \rho \vec{u}$$

Factor of γ comes from length contraction (moving charges are more closely spaced). $\vec{J}'$ comes from rod moving $-\vec{u}$ relative to S'.

- The electric and magnetic fields do not appear in a four-vector, but do appear in a "four-tensor", or matrix (signs may be wrong):

$$\mathbf{F} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

The rows and columns transform under the Lorentz transformaion, in a similar manner that rotation matrices (used to transform ordinary vector components) can be used to transform matrices. This gives transformation rules for the fields. Maxwell's equations can be written in terms of the field tensor and four-current, providing a modern proof that Maxwell's equations are invariant under the Lorentz transformation.

- Example: A particle of mass m' (unknown) decays into two particles, mass m at rest, and mass $2m$ moving at $0.5c\,\vec{\imath}$ ($\gamma = 2/\sqrt{3}$). Calculate m' and determine how fast it is moving.

$$\mathbf{p}_i = \mathbf{p}_{1f} + \mathbf{p}_{2f} = ((1+2\gamma)m, \gamma(2m)\vec{v}) = m(3.3094, 1.1547\,\vec{\imath})$$

Then

$$m' = \sqrt{(\mathbf{p}_i)^2} = 3.10m$$
$$\vec{v}_i = \vec{p}_i/E_i = (1.1547\,\vec{\imath})/(3.3094) = 0.349c\,\vec{\imath}$$

Principle of Equivalence (Ch. 15)

- Statement: There is no local experiment that can distinguish between the effects of a uniform gravitational field in an inertial frame and effects of a uniformly accelerating (non-inertial) frame.
- An enclosed chamber at rest near earth would observe the same physics as a chamber accelerating "upwards" at $9.80\,\mathrm{m/s^2}$ in free space. This is clearly true for all mechanical phenomena; Einstein extended it to all phenomena.
- Imagine an "upwardly" accelerating chamber in free space. Photon emitted from "ceiling" will require (approximately) H/c time to reach the detector at the "floor". In that time, the detector will have accelerated towards the emitter, and changed its velocity by

$$\Delta v = at = aH/c$$

The photon will be blue-shifted according to the Doppler effect:

$$\frac{\nu_{\mathrm{bot}}}{\nu_{\mathrm{top}}} = \sqrt{\frac{1+\Delta v/c}{1-\Delta v/c}} \approx 1 + \Delta v/c$$

so that

$$\frac{\Delta\nu}{\nu} \approx \frac{\Delta v}{c} = \frac{aH}{c^2}$$

A photon emitted from the floor and detected at the ceiling will be red-shifted by the same amount.

- For a chamber at rest on earth, replace a with g. A photon travelling from ceiling to floor will be blue-shifted (and floor to ceiling red-shifted) by

$$\frac{\Delta\nu}{\nu} = \frac{gH}{c^2} = \frac{\Delta\phi}{c^2}$$

  This is consistent with higher frequency photons having more kinetic energy ("falling" photons gain kinetic energy).

- Time dilation: If we imagine a clock at the ceiling emitting a wavefront every tick, then the detector at the floor will receive those ticks at a higher rate. A clock at higher altitude will be observed (by an observer standing at the lower altitude) to run at a faster rate than the same clock at lower altitude.

- Example:

$$H = 1\,\mathrm{m} \qquad g = 9.80\,\mathrm{m/s^2} \qquad \Delta\nu/\nu = 1.1 \times 10^{-16}$$

- An observer standing some distance from a black hole will observe an object falling into the black hole to take forever to cross the event horizon, and photons emitted by that object would be red-shifted to zero frequency as the object crosses the event horizon.

- In contrast, an observer crossing the event horizon will see photons from the outside blue-shifted to infinite frequency as he crosses, and will observe the entire future of the universe as the event horizon is crossed.

- Twin paradox: One twin stays in place while the other twin goes out and comes back. The twin who stays in place will be older by an amount easily calculated by time dilation. The asymmetry is due to the acceleration: During the accelerating phase, the accelerating twin is effectively at "lower altitude" than the twin staying in place, so the stationary twin's clock runs much faster.